that for an optically thin medium

$$\mathcal{I}_\nu(L) \approx \frac{h\nu}{4\pi} A_{21} N_2 S(\nu) L.$$

[Note that the spectrum of the emitted radiation is identical to the absorption lineshape $S(\nu)$.] This result is the basis of one method of measuring oscillator strengths.

(d) If $\kappa_\nu L \gg 1$ the medium is said to be *optically thick*. Show that in this case [cf. Eq. (1.2.3)]

$$\mathcal{I}_\nu(L) \approx \frac{2h\nu^3/c^2}{e^{h\nu/k_BT} - 1},$$

where T is the temperature of the medium.

(e) Discuss the evolution of the spectrum of the emitted radiation as $\kappa_\nu L$ is increased from a very small number to a very large one.

3.17. **(a)** Estimate the absorption coefficient for 589.0 nm radiation in sodium vapor containing 2.7×10^{18} atoms/m^3 at 200°C. [See J. E. Bjorkholm and A. Ashkin, *Physical Review Letters* **32**, 129 (1973)].

(b) Assuming the same conditions as in (a), plot $I_\nu(z)/I_\nu(0)$ vs. z for $\nu = \nu_0^{(2)}$, $\nu = \nu_0^{(2)} \pm \delta\nu_D$, and $\nu = \nu_0^{(2)} \pm 2\delta\nu_D$.

3.18. Show that for circularly polarized light, for which $\hat{\boldsymbol{\varepsilon}} = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$, the remarks following Eq. (3.14.1) remain correct even though (3.14.1) itself applies only to linearly polarized light.

3.19. **(a)** What is the spontaneous emission rate for the helium $1S_0$–$2P_1$ transition at 58.4 nm?

(b) A cell is filled with helium at a temperature of 300K, and the density is sufficiently low that collision broadening is negligible. Calculate the absorption coefficient for the 58.4-nm transition.

3.20. **(a)** The position vector $\mathbf{x}$ for an electron moving with velocity much less than c in a plane monochromatic wave $\hat{\boldsymbol{\varepsilon}} E_0 \cos(\omega t - kz)$ is determined by the equation of motion $m d^2\mathbf{x}/dt^2 = e\hat{\boldsymbol{\varepsilon}} E_0 \cos(\omega t - kz)$. Show that the refractive index of an electron gas is given by Eq. (3.14.13).

(b) Assume that in the ionosphere the refractive index for 100-MHz radio waves is 0.90 and that the free electrons make the greatest contribution to the index. Estimate the number density of electrons.

(c) Why is the contribution of positively charged ions to the refractive index much smaller?

(d) Choose an AM and an FM radio station in your area and compare their frequencies to the plasma frequency of the ionosphere.

3.21. Show how Eqs. (3.A.14)–(3.A.24) are altered if we do not make the assumption that $\mathbf{x}_{21} = \mathbf{x}_{12}$.

4 LASER OSCILLATION: GAIN AND THRESHOLD

4.1 INTRODUCTION

In our superficial analysis of the laser in Chapter 1 we introduced certain concepts such as gain, threshold, and feedback and indicated their importance in our understanding of lasers. We also introduced certain coefficients (a, b, f, p), which we did not derive or explain very carefully. In this chapter we will begin a detailed description of laser oscillation.

The physical system we consider is a collection of atoms (or molecules) between two mirrors. By some pumping process, such as absorption of light from a flashlamp or electron-impact excitation in a gaseous discharge, some of these atoms are promoted to excited states. The excited atoms begin radiating spontaneously, as in an ordinary fluorescent lamp. A spontaneously emitted photon can induce an excited atom to emit another photon of the same frequency and direction as the first. The more such photons are produced by stimulated emission, the faster is the production of still more photons because the stimulated emission rate is proportional to the flux of photons already in the stimulating field. (Recall the discussion in Section 3.7.)

The mirrors of the laser keep photons from escaping completely, so that they can be redirected into the active laser medium to stimulate the emission of more photons. By making the mirrors partially transmitting, some of the photons are allowed to escape. They constitute the output laser beam. The intensity of the output laser beam is determined by the rate of production of excited atoms, the reflectivities of the mirrors, and certain properties of the active atoms. We will see, in this chapter and the next, exactly how the laser output depends on these quantities.

4.2 GAIN AND FEEDBACK

In Chapter 1 the growth rate of the number of laser photons in the cavity was described by an amplification coefficient a. It is closely related to the gain coefficient derived in Section 3.12.

Consider the propagation of narrowband radiation in a medium of atoms that have a transition frequency equal, or nearly equal, to the frequency of the radiation

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Laser Physics. By Peter W. Milonni and Joseph H. Eberly

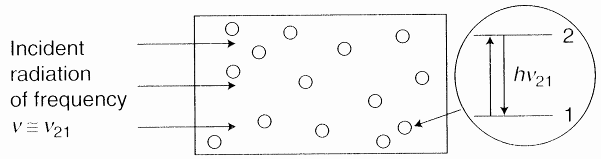

Figure 4.1 Propagation of radiation of frequency $\nu \approx \nu_{21}$ in a medium of atoms with a transition frequency ν_{21}.

(Fig. 4.1). If more atoms are in the upper level of the transition than the lower, there will be more stimulated emission than absorption, and the radiation can be amplified as it propagates. In such a case we say there is *gain* at the resonant frequency. The formula (3.12.6) for the gain coefficient shows that the same lineshape function for the medium applies for both absorption and stimulated emission. Our discussion of lineshape functions in Chapter 3 is therefore relevant to laser media as well as absorbing media. The only thing that distinguishes amplifying media from absorbing media is the sign of the *population inversion* $N_2 - N_1$ or, if the upper and lower levels have degeneracies g_2 and g_1, respectively, $N_2 - (g_2/g_1)N_1$.

Equation (3.12.10) gives an overly optimistic estimate of the growth of intensity in an amplifying ($g > 0$) medium, for it assumes that the gain is independent of intensity. As mentioned in Chapter 3 in connection with that equation, this is a valid assumption only for low intensities. In Sections 4.11 and 4.12 we will explain what it means to have a "high" intensity, and what are the implications of high intensity for the gain coefficient $g(\nu)$. For the present, however, let us accept the prediction of exponential growth as the first approximation to the actual behavior of light in an amplifier.

It is reasonable to expect that a laser can be built in the form of a pencil-shaped container of atoms for which $g > 0$ (Fig. 4.2). The consequences of such a geometry are easy to predict. Some photons are emitted along the axis of the container, where they can encounter other atoms and so induce the emission of more photons, propagating in the same direction and with the same frequency, by stimulated emission. As the number of such photons grows, the stimulated emission rate grows proportionately, so that we expect a burst of radiation to emerge from either end of the container. The direction and cross-sectional area of the beam of light so produced are determined by the container of the excited atoms.

As an example, suppose we have an amplifying medium with a gain coefficient $g = 0.01\ \text{cm}^{-1}$, an achievable gain in many laser media. With a length $L = 1$ m for the gain

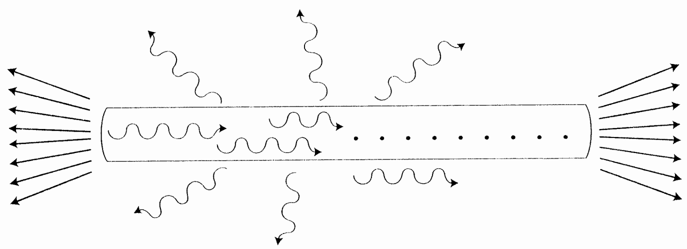

Figure 4.2 A mirrorless "laser." Photons emitted spontaneously along the axis of the tube of excited atoms are multiplied by stimulated emission, resulting in a burst of radiation.

cell, a spontaneously emitted photon at one end of the cell leads, according to Eq. (3.12.10), to an average total of

$$e^{(0.01\text{cm}^{-1})(100\text{cm})} = e^1 \approx 2.72 \tag{4.2.1}$$

photons emerging at the other end. The output of such a "laser" is obviously not very impressive.

The way to increase the photon yield from such a device is to catch photons emerging from one end and repeatedly feed them back for more amplification. In this way we can, in effect, increase the length of the gain cell. The practical way to achieve this *feedback*, of course, is to have mirrors at the ends of the container.

• It is possible, in media with very high gain, to build mirrorless (sometimes called "superradiant") lasers. The light from such a device resembles that from a conventional laser insofar as it is bright, is quasi-monochromatic, and produces a small spot on a screen. However, it does not have the same degree of temporal and spatial coherence usually associated with lasers. We discuss these coherence properties in Chapter 13. •

4.3 THRESHOLD

In a laser there is not only an increase in the number of cavity photons because of stimulated emission but also a decrease because of loss effects. These include scattering and absorption of radiation at the mirrors, as well as the "output coupling" of radiation in the form of the usable laser beam. To sustain laser oscillation the stimulated amplification must be sufficient to overcome these losses. This sets a lower limit on the gain coefficient $g(\nu)$, below which laser oscillation does not occur.

One thing we can do now is to predict, given the various losses that tend to diminish the intensity of radiation within the cavity, what minimum gain is necessary to achieve laser oscillation. The condition that the gain coefficient is greater than or equal to this lower limit is called the *threshold condition* for laser oscillation.

Ordinarily, the scattering and absorption of radiation within the gain medium of active atoms is quite small compared to the loss occurring at the mirrors of the laser. We will therefore consider in detail only the losses associated with the mirrors. Figure 4.3 shows a stylized version of a laser resonator, that is, an empty space bounded

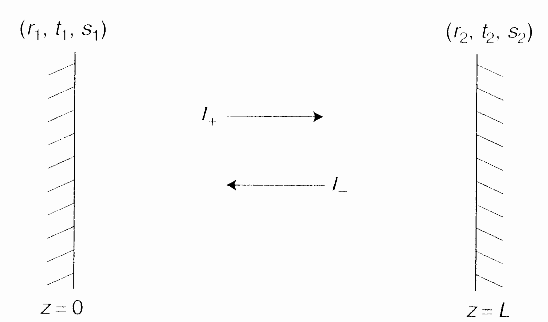

Figure 4.3 The two oppositely propagating beams in a laser cavity.

on two sides by highly reflecting mirrors. A beam of intensity I incident upon one of these mirrors is transformed into a reflected beam of intensity rI, where r is the *reflection coefficient* of the mirror. A beam of intensity tI, where t is the *transmission coefficient*, passes through the mirror. We might expect from the law of conservation of energy that

$$r + t = 1, \tag{4.3.1}$$

that is, the fraction of power reflected plus the fraction transmitted should be unity. Actually, however, some of the incident beam power may be absorbed by the mirror, tending to raise its temperature. Or some of the incident beam may be scattered away because the mirror surface is not perfectly smooth. Thus, the law of conservation of energy takes the form

$$r + t + s = 1, \tag{4.3.2}$$

where s represents the fraction of the incident beam power that is absorbed or scattered by the mirror.

Each of the mirrors of Fig. 4.3 is characterized by a set of coefficients r, t, and s. At the mirror at $z = L$ we have

$$I_\nu^{(-)}(L) = r_2 I_\nu^{(+)}(L), \tag{4.3.3a}$$

and similarly

$$I_\nu^{(+)}(0) = r_1 I_\nu^{(-)}(0) \tag{4.3.3b}$$

for the mirror at $z = 0$. Equations (4.3.3) are boundary conditions that must be satisfied by the solution of the equations describing the propagation of intensity inside the laser cavity.

What are these equations? We are now interested only in steady-state, or continuous-wave (cw), laser oscillation. Near the threshold of laser oscillation the intracavity intensity is very small, and therefore Eq. (3.12.10) is applicable. For light propagating in the positive z direction, therefore, we have

$$\frac{dI_\nu^{(+)}}{dz} = g(\nu) I_\nu^{(+)} \tag{4.3.4a}$$

near the threshold, where g may be taken to be constant. Light propagating in the negative z direction sees the same gain medium and so satisfies a similar equation (see Problem 4.1):

$$\frac{dI_\nu^{(-)}}{dz} = -g(\nu) I_\nu^{(-)}. \tag{4.3.4b}$$

The solutions of these equations are

$$I_\nu^{(+)}(z) = I_\nu^{(+)}(0) e^{g(\nu)z} \tag{4.3.5a}$$

and

$$I_\nu^{(-)}(z) = I_\nu^{(-)}(L) \exp[g(\nu)(L - z)]. \tag{4.3.5b}$$

From (4.3.5a) we see that

$$I_\nu^{(+)}(L) = I_\nu^{(+)}(0) e^{g(\nu)L} \tag{4.3.6}$$

at the right mirror ($z = L$), and the left-going beam has intensity

$$I_\nu^{(-)}(0) = I_\nu^{(-)}(L) e^{g(\nu)L} \tag{4.3.7}$$

at the left mirror ($z = 0$). In steady state the left-going beam has a fraction r_1 of itself reflected at the left mirror (at $z = 0$), and this fraction is just the right-going beam at $z = 0$. A similar consideration applies at the right mirror. Thus, we have

$$\begin{aligned} I_\nu^{(+)}(0) &= r_1 I_\nu^{(-)}(0) = r_1 \left[e^{g(\nu)L} I_\nu^{(-)}(L)\right] = r_1 e^{g(\nu)L} \left[r_2 I_\nu^{(+)}(L)\right] \\ &= r_1 r_2 e^{g(\nu)L} \left[I_\nu^{(+)}(0) e^{g(\nu)L}\right] = \left[r_1 r_2 e^{2g(\nu)L}\right] I_\nu^{(+)}(0). \end{aligned} \tag{4.3.8}$$

Similar manipulations, applied to any of the quantities $I_\nu^{(+)}(L)$, $I_\nu^{(-)}(L)$, and $I_\nu^{(-)}(0)$ lead to the same result. Therefore, if $I_\nu^{(+)}(0)$ is not zero, we must have, at steady state,

$$r_1 r_2 e^{2gL} = 1. \tag{4.3.9}$$

The steady-state value of gain that allows (4.3.9) to be satisfied is also the value at which laser action begins. For smaller values there is net attenuation of I_ν in the cavity. Thus, the value of g that satisfies (4.3.9) is labeled g_t and called the *threshold gain*:

$$g_t = \frac{1}{2L} \ln\left(\frac{1}{r_1 r_2}\right) = -\frac{1}{2L} \ln(r_1 r_2). \tag{4.3.10}$$

This expression can be rewritten usefully in the common case that $r_1 r_2 \approx 1$. Then we define $r_1 r_2 = 1 - x$, or $x = 1 - r_1 r_2$, and use the first term in the Taylor series expansion $\ln(1 - x) \approx -x$, valid when $x \ll 1$, to obtain

$$g_t = \frac{1}{2L}(1 - r_1 r_2) \quad \text{(high reflectivities)}, \tag{4.3.11}$$

which is a satisfactory approximation to (4.3.10) if $r_1 r_2 > 0.90$. The difference between (4.3.11) and (4.3.10) is connected with the assumption that the intracavity field is spatially uniform: We will see in the next chapter that spatial uniformity is a good approximation when $1 - r_1 r_2$ is small, that is, when the mirrors are highly reflecting.

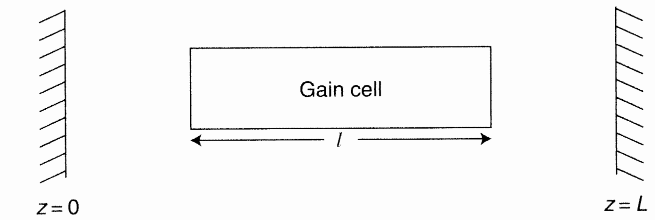

Figure 4.4 A laser in which the gain medium does not fill the entire distance L between the mirrors.

Note that if we are given the mirror reflectivities r_1 and r_2 and their separation L, and therefore the threshold gain, we can determine the population inversion necessary to achieve laser action from (3.12.6) and the atomic absorption (or stimulated emission) cross section.

Our derivation of (4.3.9) assumes that the gain medium fills the entire distance L between the mirrors. This assumption is valid for many solid-state lasers in which the ends of the gain medium are polished and coated with reflecting material. In gas and liquid lasers, however, the gain medium is usually contained in a cell of length $l < L$ that is not joined to the mirrors (Fig. 4.4). In this case the threshold condition is

$$g_t = -\frac{1}{2l}\ln(r_1 r_2) \approx \frac{1}{2l}(1 - r_1 r_2) \quad \text{(high reflectivities).} \tag{4.3.12}$$

The threshold condition (4.3.12) [or (4.3.10)] assumes that "loss" occurs only at the mirrors. This loss is associated with transmission through the mirrors, absorption by the mirrors, and scattering off the mirrors into nonlasing modes. Absorption and scattering are minimized as much as possible by using mirrors of high optical quality. Transmission, of course, is necessary if there is to be any output from the laser.

Other losses might arise from scattering and absorption within the gain medium (from nearly resonant but nonlasing transitions). Such losses are usually small, but they are not difficult to account for in the threshold condition. If a is the effective loss per unit length associated with these additional losses, then the threshold condition (4.3.12) is modified as follows:

$$g_t = -\frac{1}{2l}\ln(r_1 r_2) + a. \tag{4.3.13}$$

For our purposes these "distributed losses" (i.e., losses not associated with the mirrors) may usually be ignored.

It is instructive at this point to consider an example. A typical 632.8-nm He–Ne laser might have a gain cell of length $l = 50$ cm and mirrors with reflectivities $r_1 = 0.998$ and $r_2 = 0.980$. Thus

$$g_t = \frac{-1}{2(50)}\ln(0.998)(0.980)\,\text{cm}^{-1} = 2.2 \times 10^{-4}\,\text{cm}^{-1} \tag{4.3.14}$$

is the threshold gain.

Using this value for g_t and Eq. (3.12.6), we may calculate the *threshold population inversion* necessary to achieve lasing:

$$\Delta N_t = \left(N_2 - \frac{g_2}{g_1}N_1\right)_t = \frac{8\pi g_t}{\lambda^2 A S(\nu)} = \frac{g_t}{\sigma(\nu)}. \tag{4.3.15}$$

The A coefficient for the 632.8-nm transition in Ne is

$$A \approx 1.4 \times 10^6\,\text{s}^{-1}. \tag{4.3.16}$$

For $T \approx 400$K and the Ne atomic weight $M \approx 20$ g, we obtain from Table 3.1 the Doppler width $\delta\nu_D \approx 1500$ MHz. Thus,

$$S(\nu) \approx 6.3 \times 10^{-10}\,\text{s} \tag{4.3.17}$$

and

$$\begin{aligned}\Delta N_t &\approx \frac{(8\pi)(2.2 \times 10^{-4}\,\text{cm}^{-1})}{(6328 \times 10^{-8}\,\text{cm})^2(1.4 \times 10^6\,\text{s}^{-1})(6.3 \times 10^{-10}\,\text{s})} \\ &= 1.6 \times 10^9\,\text{atoms/cm}^3.\end{aligned} \tag{4.3.18}$$

This is a lot of atoms, but it is nevertheless quite a small number compared to the total number of Ne atoms. For a (typical) Ne partial pressure of 0.2 Torr, the total

TABLE 4.1 Quantities and Formulas Related to Gain and Threshold

The Gain Coefficient

$$g(\nu) = \frac{\lambda^2 A}{8\pi n^2}\left(N_2 - \frac{g_2}{g_1}N_1\right)S(\nu)$$

$$= \sigma(\nu)\left(N_2 - \frac{g_2}{g_1}N_1\right)$$

$\lambda = \frac{c}{\nu}$ = wavelength of radiation

A = Einstein A coefficient for spontaneous emission on the $2 \to 1$ transition

n = refractive index at wavelength λ

N_2, N_1 = number of atoms per unit volume in levels 2 and 1

g_2, g_1 = degeneracies of levels 2 and 1

$S(\nu)$ = lineshape function (Table 3.1)

Threshold Gain

$$g_t = \frac{-1}{2l}\ln(r_1 r_2) + a \approx \frac{1}{2l}(1 - r_1 r_2) + a$$

l = length of gain medium

r_1, r_2 = mirror reflectivities

a = distributed loss per unit length

number of Ne atoms per cubic centimeter is [Eq. (3.8.20)] about 4.8×10^{15}. Thus, the ratio of the threshold population inversion to the total density of atoms of the lasing species is only

$$\frac{\Delta N_t}{N} = \frac{1.6 \times 10^9}{4.8 \times 10^{15}} = \frac{1}{3} \times 10^{-6}. \tag{4.3.19}$$

Sometimes the quantity e^{gl} is called the gain, and expressed in decibels, that is,

$$G_{\mathrm{dB}} = 10 \log_{10}(e^{gl}) = 10 \log_{10}(10^{0.434gl}) = 4.34gl. \tag{4.3.20}$$

The threshold gain in our example is, thus,

$$(G_{\mathrm{dB}})_t = (4.34)(2.2 \times 10^{-4}\,\mathrm{cm}^{-1})(50\,\mathrm{cm}) = 0.048\,\mathrm{dB}. \tag{4.3.21}$$

In the laser research literature gain is usually expressed in reciprocal centimeters, although the decibel is the preferred unit in fiber optics.

In Table 4.1 we collect the formulas and terms we have used in discussing gain and threshold.

4.4 PHOTON RATE EQUATIONS

To describe time-dependent phenomena, such as pulsed laser operation or the startup of continuous-wave lasing, we must include the time derivative $\partial I_\nu/\partial t$ in the propagation equation (3.12.5). For the right- and left-going waves in the laser resonator (Fig. 4.3), we write

$$\frac{\partial I_\nu^{(+)}}{\partial z} + \frac{1}{c}\frac{\partial I_\nu^{(+)}}{\partial t} = g(\nu) I_\nu^{(+)}, \tag{4.4.1a}$$

and

$$-\frac{\partial I_\nu^{(-)}}{\partial z} + \frac{1}{c}\frac{\partial I_\nu^{(-)}}{\partial t} = g(\nu) I_\nu^{(-)}, \tag{4.4.1b}$$

respectively. Addition of these equations gives

$$\frac{\partial}{\partial z}\left[I_\nu^{(+)} - I_\nu^{(-)}\right] + \frac{1}{c}\frac{\partial}{\partial t}\left[I_\nu^{(+)} + I_\nu^{(-)}\right] = g(\nu)(I_\nu^{(+)} + I_\nu^{(-)}). \tag{4.4.2}$$

We will see in the following chapter (Sections 5.2 and 5.5) that in many lasers there is very little gross variation of $I_\nu^{(+)} - I_\nu^{(-)}$ with z. Assuming this result, we approximate (4.4.2) by the ordinary differential equation

$$\frac{d}{dt}\left[I_\nu^{(+)} + I_\nu^{(-)}\right] = cg(\nu)\left[I_\nu^{(+)} + I_\nu^{(-)}\right]. \tag{4.4.3}$$

Note that here we are assuming spatial uniformity, just as we assumed temporal uniformity (steady state) in the preceding section. In Chapter 5 we will discuss the temporal steady-state rate equation that results from a more detailed consideration of the spatial boundary conditions.

If the gain medium does not completely fill the resonator (Fig. 4.4), then $g(\nu) = 0$ outside it. If we integrate both sides of (4.4.3) over z in the region $0 < z < L$, then the left side, which is independent of z, is simply multiplied by L. However, the right side is multiplied by l, which is less than L, since $g(\nu)$ is different from zero only inside the gain medium. Thus,

$$\frac{d}{dt}\left[I_\nu^{(+)} + I_\nu^{(-)}\right] = \frac{cl}{L} g(\nu)\left[I_\nu^{(+)} + I_\nu^{(-)}\right] \tag{4.4.4}$$

is the generalization of (4.4.3) to the case $l < L$. Since the number of photons inside the cavity is proportional to the total intensity, we may also write

$$\frac{dq_\nu}{dt} = \frac{cl}{L} g(\nu) q_\nu, \tag{4.4.5}$$

where q_ν is the number of cavity photons associated with the frequency ν.

Equation (4.4.5) describes the growth in time of the number of cavity photons as a result of the absorption and induced emission of photons by the gain medium. The factor $cg(\nu)l/L$ is the growth rate. Of course, we must also consider the loss of cavity photons due to output coupling, absorption and scattering at the mirrors, and the like. We can take account of the loss associated with the output coupling of laser radiation from the cavity, which is usually the most important loss mechanism, as follows.

Radiation reflected from the mirror at $z = L$ (Fig. 4.3) has an intensity that is r_2 times the incident intensity. After it is reflected from the mirror at $z = 0$, therefore, it has an intensity $r_1 r_2$ times its intensity before the round trip inside the resonator. In other words, a fraction $1 - r_1 r_2$ of intensity is lost. Since the time it takes to make a round trip is $2L/c$, the *rate* at which intensity is lost due to the imperfect reflectivity of the mirrors is $c(1 - r_1 r_2)/2L$. In terms of photons, this loss rate is

$$\left(\frac{dq_\nu}{dt}\right)_{\text{output coupling}} = -\frac{c}{2L}(1 - r_1 r_2) q_\nu. \tag{4.4.6}$$

The total rate at which the number of cavity photons changes is therefore

$$\begin{aligned}\frac{dq_\nu}{dt} = \left(\frac{dq_\nu}{dt}\right)_{\text{gain}} + \left(\frac{dq_\nu}{dt}\right)_{\text{output coupling}} &= \frac{cl}{L} g(\nu) q_\nu - \frac{c}{2L}(1 - r_1 r_2) q_\nu \\ &= \frac{cl}{L} g(\nu) q_\nu - \frac{cl}{L} g_t q_\nu. \end{aligned}\tag{4.4.7}$$

If there are significant losses besides those occurring at the mirrors, they may be accounted for in a similar fashion. We will assume that these other losses are negligible, in which case (4.4.7) gives the rate of change with time of the number of cavity photons. Equivalently, we may write the rate equation

$$\frac{dI_\nu}{dt} = \frac{cl}{L} g(\nu) I_\nu - \frac{c}{2L}(1 - r_1 r_2) I_\nu \tag{4.4.8}$$

for the total intensity $I_\nu = I_\nu^{(+)} + I_\nu^{(-)}$ inside the cavity. If we assume equal upper and lower level degeneracies, $g_1 = g_2$, then we may write (4.4.8) as

$$\frac{dI_\nu}{dt} = \frac{cl}{L}\frac{\lambda^2 A}{8\pi}(N_2 - N_1)S(\nu)I_\nu - \frac{c}{2L}(1 - r_1 r_2)I_\nu$$
$$= \frac{cl}{L}\sigma(\nu)(N_2 - N_1)I_\nu - \frac{c}{2L}(1 - r_1 r_2)I_\nu. \tag{4.4.9}$$

4.5 POPULATION RATE EQUATIONS

The population densities N_2 and N_1 also change in time, of course, due to stimulated emission, absorption, spontaneous emission, and collisions. The effects of the first three processes on the rate equations for N_2 and N_1 are given by Eqs. (3.7.5). Collisions affecting the upper- and lower-level populations are called "inelastic" in order to distinguish them from "elastic" collisions, which do not result in a change in the energy of the colliding atoms. To account for inelastic (population-changing) collisions, we simply assert that their effect is to knock population out of levels 1 and 2 into unspecified levels at the rates Γ_1 and Γ_2. Thus, we replace Eqs. (3.7.5) by

$$\frac{dN_2}{dt} = -\Gamma_2 N_2 - \frac{\sigma(\nu)}{h\nu}I_\nu(N_2 - N_1) - A_{21}N_2, \tag{4.5.1a}$$

$$\frac{dN_1}{dt} = -\Gamma_1 N_1 + \frac{\sigma(\nu)}{h\nu}I_\nu(N_2 - N_1) + A_{21}N_2. \tag{4.5.1b}$$

We must also account for the pumping process that produces the (positive) population inversion $(N_2 - N_1)$. To do this most simply we add a term K to the population equations and call it the pumping rate into the upper level. There are several methods of arranging pumping of this kind, as discussed in Chapter 11. For the time being we simply insert a term K. With this minor modification of the population equations (4.5.1), we have the following set of coupled equations for the light and the atoms in the laser cavity:

$$\frac{dN_1}{dt} = -\Gamma_1 N_1 + A_{21}N_2 + g(\nu)\Phi_\nu, \tag{4.5.2a}$$

$$\frac{dN_2}{dt} = -(\Gamma_2 + A_{21})N_2 - g(\nu)\Phi_\nu + K, \tag{4.5.2b}$$

$$\frac{d\Phi_\nu}{dt} = \frac{cl}{L}g(\nu)\Phi_\nu - \frac{c}{2L}(1 - r_1 r_2)\Phi_\nu. \tag{4.5.2c}$$

We have used

$$I_\nu = h\nu\Phi_\nu, \tag{4.5.3}$$

where Φ_ν is the photon flux, in rewriting (4.4.8) as (4.5.2c).

For some purposes it is useful to rewrite Eqs. (4.5.2) so that they refer to absolute numbers, rather than densities, of atoms and photons. This is easy to do. The total

number of atoms in level 2 is $n_2 = N_2 V_g$, where V_g is the volume of the gain medium. Likewise the total number of atoms in the lower level of the laser transition is $n_1 = N_1 V_g$. The electromagnetic energy density u_ν in the cavity is related to intensity I_ν and photon flux Φ_ν by

$$u_\nu = \frac{I_\nu}{c} = \left(\frac{h\nu}{c}\right)\Phi_\nu, \tag{4.5.4}$$

and it is related to photon number q_ν by

$$u_\nu = \frac{h\nu q_\nu}{V},$$

where V is the cavity volume. These relations assume a uniform distribution of intensity within the cavity and a refractive index $n \approx 1$. Thus,

$$\Phi_\nu = \frac{cq_\nu}{V},$$

and Eqs. (4.5.2) may be rewritten in the form

$$\frac{dn_1}{dt} = -\Gamma_1 n_1 + A_{21}n_2 + \frac{cl}{L}g(\nu)q_\nu, \tag{4.5.5a}$$

$$\frac{dn_2}{dt} = -(\Gamma_2 + A_{21})n_2 - \frac{cl}{L}g(\nu)q_\nu + p, \tag{4.5.5b}$$

$$\frac{dq_\nu}{dt} = \frac{cl}{L}g(\nu)q_\nu - \frac{c}{2L}(1 - r_1 r_2)q_\nu, \tag{4.5.5c}$$

where we have used the relations

$$\frac{V_g}{V} = \frac{l}{L} \quad \text{and} \quad KV_g = p. \tag{4.5.6}$$

Equations (4.5.5) imply that

$$\frac{d}{dt}(n_2 + q_\nu) = -(\Gamma_2 + A_{21})n_2 + p - \frac{c}{2L}(1 - r_1 r_2)q_\nu. \tag{4.5.7}$$

This equation has an obvious interpretation. The left-hand side is the rate of change of the total number of *excitations*, that is, the number of atoms in the upper level 2 of the lasing transition plus the number of photons in the cavity. The first term on the right is the rate of decrease in the number of these excitations as a result of inelastic collisions and spontaneous emission from level 2. The second term is the rate of change associated with pumping of level 2. The last term is the rate at which excitation in the form of photons is lost from the cavity. Note that contributions from stimulated emission (or absorption) do not appear in (4.5.7) because they have canceled out: An increase in q_ν is always accompanied by an equal decrease in n_2. Further features of (4.5.7) are pointed out in Problem 4.2.

4.6 COMPARISON WITH CHAPTER 1

In Chapter 1, Section 1.5, we developed an intuitive quantum theory of the laser, introducing various rate constants in a largely ad hoc fashion. Now that we have developed rate equations for level populations and photons, it is interesting to return to this intuitive model and examine its validity.

First recall that $g(\nu) = \sigma(\nu)(N_2 - N_1)$ if the level degeneracies are equal. Thus,

$$g(\nu) \approx \sigma(\nu)N_2 = \frac{\sigma(\nu)n_2}{V_g} \tag{4.6.1}$$

if there is negligible occupation of level 1: $n_2 \gg n_1$. Now define two constant coefficients a and b:

$$a = \frac{c\sigma(\nu)l/L}{V_g} = \frac{c\sigma(\nu)}{V} \tag{4.6.2}$$

and

$$b = \frac{c}{2L}(1 - r_1 r_2). \tag{4.6.3}$$

Then Eq. (4.5.5c) for the photon number q_ν can be written in the compact form

$$\frac{dq_\nu}{dt} = an_2 q_\nu - bq_\nu, \tag{4.6.4}$$

which is exactly Eq. (1.5.1). Recall that in Chapter 1 we identified n as the number of atoms in level 2, here denoted n_2.

The equation for n_2 is easily obtained from (4.5.5b). We again invoke the assumption $n_2 \gg n_1$ to get

$$\frac{dn_2}{dt} = -an_2 q_\nu - fn_2 + p, \tag{4.6.5}$$

where

$$f = \Gamma_2 + A_{21}. \tag{4.6.6}$$

We see that Eq. (4.6.5) is the same as Eq. (1.5.2).

Thus, if the population inversion is large enough that N_1 is negligible compared to N_2, the theory developed in Chapter 1 agrees with our coupled photon-population rate equations. If N_2 is not much larger than N_1, the theory of Chapter 1 requires some minor modifications. What we were not able to do in Chapter 1, however, was to identify the constants a, b, f, and p in terms of fundamental atomic parameters like the Einstein A coefficient, the inelastic collision rate, the atomic absorption cross section, and mirror reflectivities. That has now been accomplished.

4.7 THREE-LEVEL LASER SCHEME

Thus far, we have not specified where levels 1 and 2 appear in the overall energy-level scheme of the lasing atoms. We might imagine that level 1 is the ground level and level 2 the first excited level of an atom (Fig. 4.5). When we attempt to achieve continuous laser oscillation using the two-level scheme of Fig. 4.5, however, we encounter a serious difficulty: The mechanism we use to excite atoms to level 2 can also deexcite them. For example, if we try to pump atoms from level 1 to level 2 by irradiating the medium, the radiation will induce both upward transitions $1 \longrightarrow 2$ (absorption) and downward transitions $2 \longrightarrow 1$ (stimulated emission).

As discussed in Section 4.11, the *best* we can do by this optical pumping process is to produce nearly the same number of atoms in level 2 as in level 1; we cannot obtain a positive steady-state population inversion using only two atomic levels in the pumping process.

One resolution of this difficulty is to make use of a third level, as in the *three-level laser* inversion scheme of Fig. 4.6. In such a laser, some pumping process acts between level 1 and level 3. An atom in level 3 cannot stay there forever. As a result of the pumping process, it may return to level 1, but for other reasons such as spontaneous emission or a collision with another particle, the atom may drop to a different level of lower energy. In the case of spontaneous emission the energy lost by the atom appears as radiation. In the case of collisional deexcitation, the energy lost by the atom may appear as internal excitation in a collision partner, or as an increase in the kinetic energy of the collision partners, or both. The key to the three-level inversion scheme of Fig. 4.6 is to have atoms in the pumping level 3 drop very rapidly to the upper laser level 2. This accomplishes two purposes. First, the pumping from level 1 is, in effect, directly

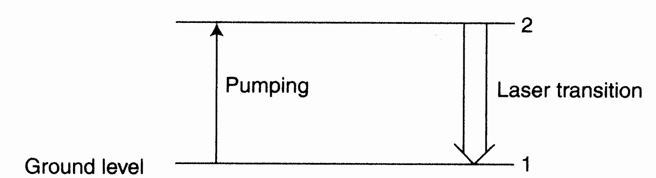

Figure 4.5 A two-level laser.

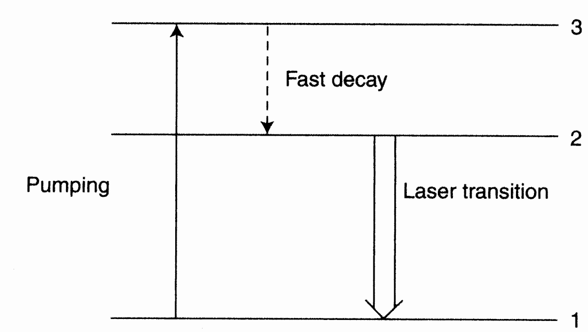

Figure 4.6 A three-level laser. Level 1 is the ground level, and laser oscillation occurs on the $2 \longrightarrow 1$ transition.

from level 1 to the upper laser level 2, because every atom finding itself in level 3 converts quickly to an atom in level 2. Second, the rapid depletion of level 3 does not give the pumping process much chance to act in reverse and repopulate the ground level 1.

We will characterize the pumping process by a rate P, so that PN_1 is the number of atoms per cubic centimeter per second that are taken from ground level 1 to level 3. Thus, the rate of change of the population N_1 of atoms per cubic centimeter in level 1 is

$$\left(\frac{dN_1}{dt}\right)_{\text{pumping}} = -PN_1 \tag{4.7.1}$$

as a result of the pumping process. Since the pumping takes atoms from level 1 to level 3, and level 3 is assumed to decay very rapidly to level 2, we may also write (see Problem 4.3)

$$\left(\frac{dN_2}{dt}\right)_{\text{pumping}} \approx \left(\frac{dN_3}{dt}\right)_{\text{pumping}} = -\left(\frac{dN_1}{dt}\right)_{\text{pumping}} = PN_1 \tag{4.7.2}$$

for the rate of change of population of level 2 due to pumping.

Atoms in level 2 can decay, by spontaneous emission or via collisions, as indicated in population Eq. (4.5.2b) or (4.5.5b). For simplicity we will now assume that level 2 decays only into level 1 by these processes, and we will denote the rate by Γ_{21}. That is, we assume

$$\left(\frac{dN_2}{dt}\right)_{\text{decay}} = -\Gamma_{21}N_2, \qquad \left(\frac{dN_1}{dt}\right)_{\text{decay}} = \Gamma_{21}N_2, \tag{4.7.3}$$

for the population changes associated with the decay of level 2. The total rates of change of the populations of levels 1 and 2 are therefore

$$\frac{dN_1}{dt} = -PN_1 + \Gamma_{21}N_2 + \sigma\Phi_\nu(N_2 - N_1), \tag{4.7.4a}$$

$$\frac{dN_2}{dt} = PN_1 - \Gamma_{21}N_2 - \sigma\Phi_\nu(N_2 - N_1). \tag{4.7.4b}$$

Equations (4.7.4) imply the conservation law

$$\frac{d}{dt}(N_1 + N_2) = 0,$$

or

$$N_1 + N_2 = \text{const} = N_T. \tag{4.7.5}$$

By ignoring any other atomic energy levels, and assuming that level 3 decays practically instantaneously into level 2, we are assuming that each active atom of the gain medium must be either in level 1 or level 2. Therefore, the conserved quantity N_T is simply the total number of active atoms per unit volume.

We can now draw some important conclusions about the "threshold region" of steady-state (cw) laser oscillation. Near threshold the number of cavity photons is small enough that stimulated emission may be omitted from Eqs. (4.7.4). In particular, we can determine from these equations the threshold pumping rate necessary to achieve a population inversion, together with the threshold power expended in the process.

In the steady state N_1 and N_2 are not changing in time. The steady-state values $\overline{N}_1$ and $\overline{N}_2$, therefore, satisfy Eqs. (4.7.4) with $dN_1/dt = dN_2/dt = 0$. Thus, if Φ_ν is so small that the last terms in (4.7.4) are negligible, we find

$$\overline{N}_2 = \frac{P}{\Gamma_{21}}\overline{N}_1 \tag{4.7.6}$$

in the steady state. Since (4.7.5) must hold for all possible values of N_1 and N_2, including the steady-state values $\overline{N}_1$ and $\overline{N}_2$, we also have

$$\overline{N}_1 + \overline{N}_2 = N_T. \tag{4.7.7}$$

Equations (4.7.6) and (4.7.7) may be solved for $\overline{N}_1$ and $\overline{N}_2$ to obtain

$$\overline{N}_1 = \frac{\Gamma_{21}}{P + \Gamma_{21}}N_T \tag{4.7.8a}$$

and

$$\overline{N}_2 = \frac{P}{P + \Gamma_{21}}N_T. \tag{4.7.8b}$$

The steady-state threshold-region population inversion is therefore

$$\overline{N}_2 - \overline{N}_1 = \frac{P - \Gamma_{21}}{P + \Gamma_{21}}N_T. \tag{4.7.9}$$

To have a positive steady-state population inversion, and therefore a positive gain, we must obviously have

$$P > \Gamma_{21}, \tag{4.7.10}$$

which simply says that the pumping rate into the upper laser level must exceed the decay rate. The greater the pumping rate with respect to the decay rate, the greater the population inversion and gain.

The pumping of an atom from level 1 to level 3 requires an energy

$$E_3 - E_1 = h\nu_{31}. \tag{4.7.11}$$

The power per unit volume delivered to the active atoms in the pumping process is therefore

$$\frac{\text{Pwr}}{V} = h\nu_{31}P\overline{N}_1 \tag{4.7.12}$$

in the steady state. Using (4.7.8), we may write this as

$$\frac{\text{Pwr}}{V} = \frac{h\nu_{31}P\Gamma_{21}}{P + \Gamma_{21}}N_T. \tag{4.7.13}$$

Now from (4.7.10) we may regard

$$P_{\min} = \Gamma_{21} \tag{4.7.14}$$

as the minimum pumping rate necessary to reach positive gain. Substituting $P_{\min}$ for P in (4.7.13), we obtain

$$\left(\frac{\text{Pwr}}{V}\right)_{\min} = \frac{1}{2}\Gamma_{21}N_T h\nu_{31} \tag{4.7.15}$$

as the minimum power per unit volume that must be exceeded to produce a positive gain. With this amount of pumping power delivered to the active medium, we see from (4.7.8) (with $P = P_{\min} = \Gamma_{21}$) that half the active atoms are in the lower level of the laser transition and half are in the upper level. A pumping power density greater than (4.7.15) makes $\overline{N}_2 > \overline{N}_1$.

4.8 FOUR-LEVEL LASER SCHEME

Another useful model for achieving population inversion is the *four-level laser* scheme shown in Fig. 4.7. Pumping proceeds from the ground level 0 to the level 3, which, as in the three-level laser, decays rapidly into the upper laser level 2. In this model the lower laser level 1 is not the ground level, but an excited level that can itself decay into the ground level. This represents an advantage over the three-level laser, for the depletion of the lower laser level obviously enhances the population inversion on the laser transition. That is, a decrease in N_1 results in an increase in $N_2 - N_1$.

As in a three-level laser the decay from level 3 to level 2 is ideally instantaneous, that is, extremely rapid compared to any other rates in the population rate equations. Then we may take $N_3 \approx 0$, and the population rate equations for the four-level laser take the form

$$\frac{dN_0}{dt} = -PN_0 + \Gamma_{10}N_1, \tag{4.8.1a}$$

$$\frac{dN_1}{dt} = -\Gamma_{10}N_1 + \Gamma_{21}N_2 + \sigma(\nu)(N_2 - N_1)\Phi_\nu, \tag{4.8.1b}$$

$$\frac{dN_2}{dt} = PN_0 - \Gamma_{21}N_2 - \sigma(\nu)(N_2 - N_1)\Phi_\nu, \tag{4.8.1c}$$

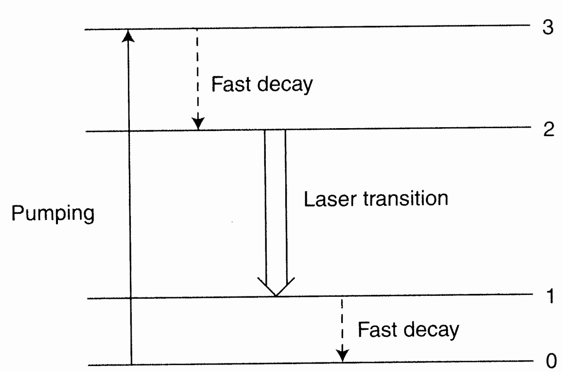

Figure 4.7 A four-level laser. The lower laser level 1 decays into the ground level 0.

where P is again the pumping rate out of the ground level 0, and PN_0 is the upper level pumping rate denoted by K in (4.5.2). Γ_{21} and Γ_{10} are the rates for the decay processes $2 \to 1$ and $1 \to 0$, respectively, and we have made the same approximation (4.7.2) for the pumping process as in the three-level case. Note that Eqs. (4.8.1) imply the conservation law

$$N_0 + N_1 + N_2 = \text{const} = N_T. \tag{4.8.2}$$

If the stimulated emission rate is very small compared to the pumping and decay rates, Eqs. (4.8.1) give the steady-state populations

$$\overline{N}_0 = \frac{\Gamma_{10}\Gamma_{21}}{\Gamma_{10}\Gamma_{21} + \Gamma_{10}P + \Gamma_{21}P}N_T, \tag{4.8.3a}$$

$$\overline{N}_1 = \frac{\Gamma_{21}P}{\Gamma_{10}\Gamma_{21} + \Gamma_{10}P + \Gamma_{21}P}N_T, \tag{4.8.3b}$$

$$\overline{N}_2 = \frac{\Gamma_{10}P}{\Gamma_{10}\Gamma_{21} + \Gamma_{10}P + \Gamma_{21}P}N_T. \tag{4.8.3c}$$

The steady-state population inversion of the laser transition is therefore (Problem 4.4)

$$\overline{N}_2 - \overline{N}_1 = \frac{P(\Gamma_{10} - \Gamma_{21})N_T}{\Gamma_{10}\Gamma_{21} + \Gamma_{10}P + \Gamma_{21}P}. \tag{4.8.4}$$

Thus, the pumped ($P \neq 0$) four-level system will always have a steady-state population inversion when

$$\Gamma_{10} > \Gamma_{21}, \tag{4.8.5}$$

that is, when the lower laser level decays more rapidly than the upper laser level. When we have

$$\Gamma_{10} \gg \Gamma_{21}, P, \tag{4.8.6}$$

then $N_0 \approx N_T$, $N_1 \approx 0$ and Eq. (4.8.4) reduces to

$$\overline{N}_2 - \overline{N}_1 \approx \overline{N}_2 \approx \frac{P}{P + \Gamma_{21}}N_T. \tag{4.8.7}$$

4.9 PUMPING THREE- AND FOUR-LEVEL LASERS

It is interesting to compare the pumping rates necessary for laser oscillation in the three- and four-level lasers. To achieve laser oscillation, we must produce a gain larger than the threshold value g_t, and therefore a population inversion greater than ΔN_t, where

$$g_t \equiv \sigma(\nu)\,\Delta N_t \tag{4.9.1}$$

is the threshold gain for frequency ν. Setting $\overline{N}_2 - \overline{N}_1$ equal to the threshold inversion in (4.7.9), we obtain

$$(P_t)_{\text{three-level laser}} = \frac{N_T + \Delta N_t}{N_T - \Delta N_t}\Gamma_{21} \tag{4.9.2}$$

for the threshold pumping rate for laser oscillation in the three-level laser. From (4.8.7), on the other hand, the pumping rate necessary for a four-level laser satisfying (4.8.6) is

$$(P_t)_{\text{four-level laser}} = \frac{\Delta N_t}{N_T - \Delta N_t}\Gamma_{21} \tag{4.9.3}$$

The ratio of (4.9.3) to (4.9.2) is

$$\frac{(P_t)_{\text{four-level laser}}}{(P_t)_{\text{three-level laser}}} = \frac{\Delta N_t}{N_T + \Delta N_t}. \tag{4.9.4}$$

Ordinarily, the population inversion (at threshold or otherwise) is very small compared to the total number of active atoms; recall our example near the end of Section 4.3. From (4.9.4), therefore, we see that

$$(P_t)_{\text{four-level laser}} \ll (P_t)_{\text{three-level laser}}. \tag{4.9.5}$$

Thus, a much larger pumping rate is necessary to achieve laser oscillation in a three-level laser than in a four-level laser.

In the three-level laser the pumping power density necessary to establish the threshold inversion ΔN_t is

$$\left(\frac{(\text{Pwr})_t}{V}\right)_{\text{three-level laser}} = h\nu_{31}(N_1)_t(P_t)_{\text{three-level laser}}, \tag{4.9.6}$$

where $(N_1)_t$ is given by (4.7.8a) with $P = (P_t)_{\text{three-level laser}}$. Assuming $\Delta N_t \ll N_T$, we conclude from (4.9.2) that

$$(P_t)_{\text{three-level laser}} \approx \Gamma_{21}, \tag{4.9.7}$$

and from (4.7.8) that

$$(N_1)_t \approx \frac{N_T}{2}. \tag{4.9.8}$$

Equation (4.9.6) therefore becomes

$$\left(\frac{(\text{Pwr})_t}{V}\right)_{\text{three-level laser}} \approx \frac{1}{2}h\nu_{31}N_T\Gamma_{21}, \tag{4.9.9}$$

which, because of the approximation $\Delta N_t \ll N_T$, is the same as (4.7.15). Thus, when $\Delta N_t \ll N_T$ the minimum pumping power density necessary to achieve positive gain is approximately the same as that necessary to reach threshold and laser oscillation. In the four-level case, on the other hand, we obtain

$$\left(\frac{(\text{Pwr})_t}{V}\right)_{\text{four-level laser}} \approx h\nu_{30}\,\Delta N_t\,\Gamma_{21} \tag{4.9.10}$$

for $\Delta N_t \ll N_T$. The ratio of (4.9.10) to (4.9.9) is

$$\frac{[(\text{Pwr})_t/V]_{\text{four-level laser}}}{[(\text{Pwr})_t/V]_{\text{three-level laser}}} \approx \frac{2\nu_{30}\,\Delta N_t}{\nu_{31}N_T} \tag{4.9.11}$$

if we take the upper-laser-level decay rate Γ_{21} to be the same in the two cases. This shows again that, other things being roughly commensurate, much less power is required to achieve laser oscillation in the four-level case.

4.10 EXAMPLES OF THREE- AND FOUR-LEVEL LASERS

Real lasers seldom fit very neatly into our three- and four-level categories. However, these idealizations sometimes provide a useful framework for rough estimates of pump power requirements. We will illustrate this for cw ruby and Nd : YAG solid-state lasers, which are approximately three-level and four-level systems, respectively.

Ruby was the gain medium for the very first laser, but the ruby laser has largely been replaced by other, more efficient solid-state lasers. It nevertheless illustrates nicely the concept of a three-level laser. As mentioned in Section 3.1, ruby is the crystal Al_2O_3 with chromium ions (Cr^{3+}) replacing some of the aluminum ions (Al^{3+}); the concentration of chromium is only about 0.05% by weight. The relevant energy levels for the ruby laser are those of the Cr^{3+} ion in the host crystal lattice. The laser is *optically pumped*, that is, a population inversion is obtained by the absorption of radiation from another laser or a lamp, typically a high-pressure Xe or Hg flashlamp (Section 11.12). It is approximately a three-level laser, although the third "level" really consists of two broad bands of energy, both decaying rapidly (rate $\approx 10^7\ \text{s}^{-1}$) into the upper laser level 2. At room temperature the decay rate of the upper laser level is

$$\Gamma_{21} \approx \tfrac{1}{2} \times 10^3\ \text{s}^{-1}. \tag{4.10.1}$$

The density of "active atoms" (i.e., Cr^{3+} ions) is (Problem 4.5)

$$N_T \approx 1.6 \times 10^{19}\ \text{cm}^{-3}. \tag{4.10.2}$$

The excitation energy required from the pump, the energy difference between the ground level and the lower pump band, is about 2.25 eV, corresponding to a wavelength of about 550 nm (green). From (4.9.9), therefore, the minimum pumping power density necessary to achieve nonnegative gain [and also, according to (4.7.15), approximately the pumping power density necessary for laser oscillation] is

$$\left(\frac{\text{Pwr}}{V}\right)_{\min} \approx \frac{1}{2}\left(\frac{1}{2} \times 10^3\ \text{s}^{-1}\right)(1.6 \times 10^{19}\ \text{cm}^{-3})(2.25\ \text{eV}) \approx 1\ \text{kW/cm}^3. \tag{4.10.3}$$

For a 5-cm-long ruby rod of radius 2 mm, the required pump power is

$$\text{Pwr} \approx \left(\frac{1\ \text{kW}}{\text{cm}^3}\right)\pi\,(0.2\ \text{cm})^2(5\ \text{cm}) = 600\ \text{W}. \tag{4.10.4}$$

This is a rather large amount of power. In fact, much *more* power is actually required to operate a cw ruby laser because (4.10.4) only gives the power that must actually be absorbed by the Cr^{3+} ions. In reality only a small fraction (typically $\approx 0.1\%$) of the electrical power delivered to the lamp is converted to useful laser radiation.

It is also interesting to estimate the population inversion necessary for threshold gain in a typical ruby laser. At room temperature the 694.3-nm laser transition has a Lorentzian lineshape of width (HWHM)

$$\delta\nu_0 \approx 170\,\text{GHz}, \tag{4.10.5}$$

and the A coefficient for spontaneous emission is

$$A \approx 230\,\text{s}^{-1}. \tag{4.10.6}$$

The line-center cross section for stimulated emission is therefore

$$\sigma = \frac{\lambda^2 A}{8\pi n^2}\frac{1}{\pi\delta\nu_0} \approx 2.7 \times 10^{-20}\,\text{cm}^2, \tag{4.10.7}$$

where we have used the value $n = 1.76$ for the refractive index of ruby. If we assume a resonator with mirror reflectivities $r_1 = 1.0$ and $r_2 = 0.96$, and a scattering loss of 3% per round-trip pass through the gain cell, then the threshold gain for laser oscillation is [Eq. (4.3.13)]

$$g_t = \frac{-1}{2(5\,\text{cm})}\ln(0.96) + \frac{0.03}{2(5\,\text{cm})} = 7.1 \times 10^{-3}\,\text{cm}^{-1} \tag{4.10.8}$$

for a ruby rod 5 cm long. From (4.3.15), (4.10.7), and (4.10.8), therefore, we calculate a population inversion threshold

$$\Delta N_t \approx \frac{7.1 \times 10^{-3}\,\text{cm}^{-1}}{2.7 \times 10^{-20}\,\text{cm}^2} = 2.6 \times 10^{17}\,\text{cm}^{-3}. \tag{4.10.9}$$

This is much larger than the sort of population inversion necessary for a typical He–Ne laser [Eq. (4.3.18)]. The difference stems from the much larger stimulated emission cross section of the 632.8-nm laser transition of Ne, which in turn results from the much larger A coefficient and much smaller linewidth than in ruby. This illustrates an important point: Gas lasers obviously have a much smaller density of atoms than solid-state lasers, but this does not necessarily mean that they have smaller gains. In fact, many of the most powerful lasers are gas lasers. The reasons for this are discussed in Chapter 11.

In the 1.06-μm (1064-nm) Nd : YAG laser, the active atoms are also impurities in a crystal lattice, in this case Nd^{3+} ions in yttrium aluminum garnet ($Y_3Al_5O_{12}$, called YAG). The Nd : YAG laser is approximately a four-level system, with upper-level decay rate

$$\Gamma_{21} \approx 4400\,\text{s}^{-1} \tag{4.10.10}$$

and stimulated emission cross section[1]

$$\sigma \approx 3 \times 10^{-19}\,\text{cm}^2 \tag{4.10.11}$$

at room temperature. If we assume the same threshold gain (4.10.8) as in our calculation for the ruby laser, we obtain a population inversion threshold

$$\Delta N_t \approx \frac{7.1 \times 10^{-3}\,\text{cm}^{-1}}{3 \times 10^{-19}\,\text{cm}^2} \approx 2 \times 10^{16}\,\text{cm}^{-3}, \tag{4.10.12}$$

which, because of the relatively large stimulated emission cross section for Nd^{3+}, is considerably smaller than the value (4.10.9) for ruby.

The pump "level 3" for the Nd : YAG laser is actually a series of energy bands located between about 1.63 and 3.13 eV above the ground level. If we take the energy difference $E_3 - E_0$ in our four-level model (Fig. 4.7) to be the average value, 2.38 eV, we obtain from (4.9.10) the pumping power density for threshold:

$$\frac{(\text{Pwr})_{\text{min}}}{V} \approx 30\,\text{W/cm}^3. \tag{4.10.13}$$

For a 5-cm Nd : YAG rod of radius 2 mm, therefore, the threshold pump power is

$$\text{Pwr} \approx 20\,\text{W}, \tag{4.10.14}$$

much smaller than the estimate (4.10.4) for ruby.

4.11 SATURATION

We remarked in Sections 3.12 and 4.2 that exponential growth of intensity in a gain medium is only an approximation, and that the approximation breaks down when the intensity is sufficiently large. Exponential attenuation in an absorbing medium is likewise a low-intensity approximation.

To understand this, let us return to the rate equations (3.7.5) for the populations of two nondegenerate levels. No upper-level pumping processes are included in these equations, only absorption and stimulated and spontaneous emission. We further assume that the intensity I_ν of the field is constant in time. The steady-state solutions $\overline{N}_2$ and $\overline{N}_1$ obtained by setting the derivatives equal to zero are easily found to be

$$\overline{N}_2 = \frac{\sigma(\nu)I_\nu/h\nu}{A_{21} + 2\sigma(\nu)I_\nu/h\nu}N = \frac{\frac{1}{2}I_\nu/I_\nu^{\text{sat}}}{1 + I_\nu/I_\nu^{\text{sat}}}N, \tag{4.11.1a}$$

$$\overline{N}_1 = \frac{A_{21} + \sigma(\nu)I_\nu/h\nu}{A_{21} + 2\sigma(\nu)I_\nu/h\nu}N = \frac{1 + \frac{1}{2}I_\nu/I_\nu^{\text{sat}}}{1 + I_\nu/I_\nu^{\text{sat}}}N, \tag{4.11.1b}$$

[1]There are significant differences in reported measurements of these parameters for Nd : YAG, and the estimates used here should be considered reliable only to within about a factor of 2.

where $N = N_1 + N_2 = \overline{N}_1 + \overline{N}_2$ and

$$I_\nu^{\text{sat}} = \frac{h\nu A_{21}}{2\sigma(\nu)} \tag{4.11.2}$$

is the *saturation intensity*. The absorption coefficient is then

$$a(\nu) = \sigma(\nu)(\overline{N}_1 - \overline{N}_2) = \frac{a_0(\nu)}{1 + I_\nu/I_\nu^{\text{sat}}}, \tag{4.11.3}$$

where

$$a_0(\nu) = \sigma(\nu)N \tag{4.11.4}$$

is the *small-signal absorption coefficient*, the absorption coefficient when the intensity I_ν is small compared to I_ν^{sat}. In this case $\overline{N}_1 \approx N$ and $\overline{N}_2 \approx 0$, that is, practically all the atoms are in the ground level 1.

As I_ν/I_ν^{sat} increases, the absorption coefficient "saturates," becoming smaller and smaller as I_ν increases. For $I_\nu \gg I_\nu^{\text{sat}}$, $\overline{N}_2 \approx \overline{N}_1 \approx N/2$. In this strongly saturated regime the (equal) rates of absorption and stimulated emission are so large that the atoms are equally likely to be found in the excited level as the ground level. The larger I_ν^{sat}, the larger the field intensity I_ν has to be to produce significant saturation of the transition. Saturation of an absorbing transition arises from the excitation of the upper level, which increases stimulated emission and reduces the absorption.

As discussed in the following section, the gain coefficient of an amplifying medium exhibits essentially the same saturation behavior. In this case the saturation arises from the growth due to stimulated emission of the lower-level population, which enhances absorption and thereby reduces the amplification of the field. The dependence of $g(\nu)$ on I_ν means that the solution of Eq. (3.12.9) is not the simple exponentially growing intensity (3.12.10). The correct solution, which is given in the following chapter, grows exponentially with z only as long as I_ν is small compared to I_ν^{sat}. The exponential attenuation in an absorber is likewise a valid approximation only for intensities small compared to I_ν^{sat}. The saturation intensity, whose numerical value is determined by the transition cross section and rates, thus provides the measure of whether a given field intensity is "large" or "small" in terms of its ability to saturate the transition.

A different, somewhat more restrictive interpretation of saturation is possible. Consider a homogeneously broadened transition having a Lorentzian lineshape of width $\delta\nu_0$. After some simple algebra, using Eqs. (4.11.2), (4.11.3), and (3.7.4), we find that

$$a(\nu) = \frac{\lambda^2 A_{21}}{8\pi} N \frac{(1/\pi)\delta\nu_0}{(\nu - \nu_0)^2 + \delta\nu_0'^2} = \frac{a_0(\nu_0)\delta\nu_0^2}{(\nu - \nu_0)^2 + \delta\nu_0'^2}, \tag{4.11.5}$$

where we define

$$\delta\nu_0' = \delta\nu_0 \sqrt{1 + I_\nu/I_{\nu_0}^{\text{sat}}}. \tag{4.11.6}$$

We see that, in effect, the width $\delta\nu_0$ of the transition is increased by the factor $\sqrt{1 + I_\nu/I_{\nu_0}^{\text{sat}}}$. In other words, we can interpret the saturation of the transition with increasing intensity as an effective "power broadening" of the linewidth.

Saturation will always occur at sufficiently high intensities, regardless of whether the transition is homogeneously or inhomogeneously broadened. The saturation intensity (4.11.2), because of its dependence on $\sigma(\nu)$, will vary with the lineshape function $S(\nu)$. For a Doppler-broadened transition, for instance,

$$I_{\nu_0}^{\text{sat}} = \frac{h\nu A_{21}}{2(\lambda^2 A_{21}/8\pi)S(\nu_0)} = \frac{4\pi^2 hc}{\lambda^3} \delta\nu_D \sqrt{\frac{1}{4\pi \ln 2}} \tag{4.11.7}$$

at line center ($\nu = \nu_0$), whereas for a transition with a Lorentzian lineshape (3.4.26),

$$I_{\nu_0}^{\text{sat}} = \frac{4\pi^2 hc}{\lambda^3} \delta\nu_0. \tag{4.11.8}$$

These formulas are based on the assumption that spontaneous emission is the only (intensity-independent) decay process for the upper level of the transition, and that the lower level is the ground level of the atom. In general the saturation intensity will depend on both upper- and lower-level decay rates associated with collisional as well as radiative processes, and it can also depend on the level degeneracies. Here we are less interested in the detailed form of I_ν^{sat} as we are in the fact that the absorption and gain coefficients saturate, in many situations of practical interest, as $1/[1 + I_\nu/I_\nu^{\text{sat}}]$, whatever the form of I_ν^{sat}. In the following section we will derive saturation formulas specifically for the gain coefficient of our idealized three- and four-level lasers.

Although they account only for radiative excitation and deexcitation processes, the formulas obtained here for I_ν^{sat} are nevertheless useful in their own right. Consider as an example the absorption by sodium vapor of radiation resonant with the $3S_{1/2}(F = 2) \leftrightarrow 3P_{3/2}$ transition: $\nu = \nu_0^{(2)}$ in the notation of Section 3.13. For Doppler broadening at $T = 200$K, we calculated in Section 3.13 the Doppler width $\delta\nu_D = 1$ GHz. Then, from (4.11.7),

$$I_{\nu_0}^{\text{sat}} \approx 1.3\,\text{W/cm}^2. \tag{4.11.9}$$

If instead we assume radiative broadening, for which $\delta\nu_0 = A_{21}/4\pi$ (Section 3.11), then

$$I_{\nu_0}^{\text{sat}} \approx \frac{\pi hc}{\lambda^3} A_{21} = 19\,\text{mW/cm}^2, \tag{4.11.10}$$

where we have used $A_{21} = 6.2 \times 10^7\,\text{s}^{-1}$ for the spontaneous emission rate of the sodium D_2 line (Section 3.13). These results do not account for the level degeneracies and hyperfine structure, and thus do not include factors such as g_2/g_1 or $\frac{5}{8}$ or $\frac{3}{8}$ appearing in Eq. (3.13.9). However, because these omissions give rise only to factors of order unity, the numerical values for I_ν^{sat} are of the correct magnitude. The large disparity in these two saturation intensities is not unusual; saturation intensities can vary widely for the same absorbing or emitting atoms, depending on the physical situation.

• Saturation of an atomic transition has been observed rather directly in experiments using a sodium beam. Well-collimated atomic beams are formed by those atoms that have passed from an oven (used to produce a vapor) through two (or more) successive pinholes. Irradiation by a laser beam propagating at a right angle to the atomic beam nearly eliminates any Doppler broadening and results, typically, in purely radiative broadening of the resonant transition. By monitoring the intensity of the spontaneously emitted radiation, one can infer the dependence of the excited-state population on the laser intensity or frequency.

As discussed in Section 14.3, it is possible to "align" atoms by irradiating them with polarized light. For instance, if a sodium beam is irradiated with circularly polarized laser radiation, it can be "aligned" such that only transitions between the two states $3S_{1/2}(F=2, M=2)$ and $3P_{3/2}(F=3, M=3)$ are possible. For this transition the saturation intensity $I_{\nu_0}^{\text{sat}}$ can be shown to be $\pi hcA_{21}/3\lambda^3$, that is, 6.3 mW/cm^2, or one third the value given by Eq. (4.11.10), which assumes no alignment (Problem 14.8).

The FWHM radiative linewidth of the sodium D_2 line is [Eq. (3.11.2)] $2\delta\nu_0 = A_{21}/2\pi =$ 10 MHz. According to (4.11.6), therefore, the power-broadened radiative linewidth (FWHM) of the $3S_{1/2}(F=2, M=2) \leftrightarrow 3P_{3/2}(F=3, M=3)$ transition should be

$$\delta\nu_0' \approx 10\sqrt{1 + I_\nu/6.3}\ \text{MHz}, \tag{4.11.11}$$

where I_ν is the laser intensity in units of mW/cm^2. Measurements of $\delta\nu_0'$ for $I_\nu = 0.84$, 3.5, 90, and 170 mW/cm^2 gave $\delta\nu_0' = 12.4 \pm 0.8$, 13.8 ± 0.9, 41.2 ± 1.8, and 53.7 ± 2.8 MHz, respectively, in good agreement with the variation predicted by (4.11.11).[2] The dependence of the scattered intensity on the laser intensity at resonance, similarly, was found to be well described by the factor $1/[1 + I_\nu/I_{\nu_0}^{\text{sat}}]$. •

4.12 SMALL-SIGNAL GAIN AND SATURATION

Equation (4.7.9) gives the steady-state population inversion for a three-level laser when the stimulated emission rate is negligible. In general, of course, the stimulated emission rate is not negligible, and here we consider the steady-state population inversion in the more general case. For this we require the steady-state solutions of Eqs. (4.7.4). These may be obtained by noting that the following replacements for P and Γ_{21} (in the equations *without* stimulated emission),

$$P \longrightarrow P + \sigma\Phi_\nu, \tag{4.12.1a}$$

$$\Gamma_{21} \longrightarrow \Gamma_{21} + \sigma\Phi_\nu, \tag{4.12.1b}$$

are sufficient to reinstate all stimulated-emission terms. Here Φ_ν is the steady-state (i.e., time-independent) cavity photon flux. Thus, the steady-state solutions of (4.7.4) may be obtained by making the same replacements in the solutions (4.7.8). Likewise the steady-state population inversion $\overline{N}_2 - \overline{N}_1$ in the general case follows when the replacements (4.12.1) are made in (4.7.9):

$$\overline{N}_2 - \overline{N}_1 = \frac{(P - \Gamma_{21})N_T}{P + \Gamma_{21} + 2\sigma\Phi_\nu}. \tag{4.12.2}$$

This is the generalization of (4.7.9) to the case in which the stimulated emission rate is not negligible compared to P and Γ_{21}.

[2]M. L. Citron, H. R. Gray, C. W. Gabel, and C. R. Stroud, Jr., *Physical Review A* **16**, 1507 (1977).

The steady-state gain coefficient for a three-level laser follows from (3.12.6) and (4.12.2). Assuming $g_1 = g_2$, we have

$$\begin{aligned} g(\nu) &= \frac{\sigma(\nu)(P - \Gamma_{21})N_T}{P + \Gamma_{21} + 2\sigma(\nu)\Phi_\nu} = \frac{\sigma(\nu)(P - \Gamma_{21})N_T}{P + \Gamma_{21}} \frac{1}{1 + [2\sigma(\nu)\Phi_\nu/(P + \Gamma_{21})]} \\ &= \frac{g_0(\nu)}{1 + \Phi_\nu/\Phi_\nu^{\text{sat}}} = \frac{g_0(\nu)}{1 + I_\nu/I_\nu^{\text{sat}}}, \end{aligned} \tag{4.12.3}$$

where we define the *small-signal gain*

$$g_0(\nu) = \frac{\sigma(\nu)(P - \Gamma_{21})N_T}{P + \Gamma_{21}} \tag{4.12.4}$$

and the *saturation flux*

$$\Phi_\nu^{\text{sat}} = \frac{P + \Gamma_{21}}{2\sigma(\nu)}. \tag{4.12.5}$$

The corresponding expressions for the saturation intensity and photon number are (see Problem 4.7)

$$I_\nu^{\text{sat}} = h\nu\Phi_\nu^{\text{sat}} = \frac{h\nu(P + \Gamma_{21})}{2\sigma(\nu)} \tag{4.12.6}$$

and

$$q_\nu^{\text{sat}} = \frac{V}{c}\Phi_\nu^{\text{sat}} = \frac{P + \Gamma_{21}}{2c\sigma(\nu)}V. \tag{4.12.7}$$

The gain coefficient for the three-level laser, therefore, saturates in the same way as the absorption coefficient (4.11.3) of a two-level transition. The saturation intensities in the two cases are different; in particular, I_ν^{sat} for the three-level laser depends not only on $h\nu/\sigma(\nu)$ but also on the pumping rate P and the decay rate Γ_{21}.

For $I_\nu \ll I_\nu^{\text{sat}}$, $g(\nu) \approx g_0(\nu)$, which, of course, is why $g_0(\nu)$ is called the "small-signal" gain coefficient. The maximum gain is $g_0(\nu_0)$, that is, the gain when $I_\nu \ll I_\nu^{\text{sat}}$ and the field frequency matches the line-center frequency ν_0, where $\sigma(\nu_0)$ has its maximum value. When the lineshape is Lorentzian, with HWHM width $\delta\nu_0$, we have

$$g(\nu) = g_0(\nu_0)\frac{1}{(\nu_0 - \nu)^2/\delta\nu_0^2 + 1 + (\Phi_\nu/\Phi_{\nu_0}^{\text{sat}})}. \tag{4.12.8}$$

The cavity frequencies at which there is small-signal gain sufficient to overcome loss in a laser are generally those within about $\delta\nu_0$ of line center ($\nu = \nu_0$); $\delta\nu_0$ can be called the small-signal *gain bandwidth*. In Section 1.3 we showed by way of an example how the gain bandwidth and the cavity mode spacing together determine the number of possible frequencies that can lase.

In Fig. 4.8 we plot $g(\nu)$ vs. ν as given in (4.12.8) for several values of Φ_ν and $g(\nu)$ vs. Φ_ν for several values of ν. Clearly, it is harder to saturate $g(\nu)$ away from line center. Alternatively, for higher fluxes the halfwidth of $g(\nu)$ is greater. This is exactly the

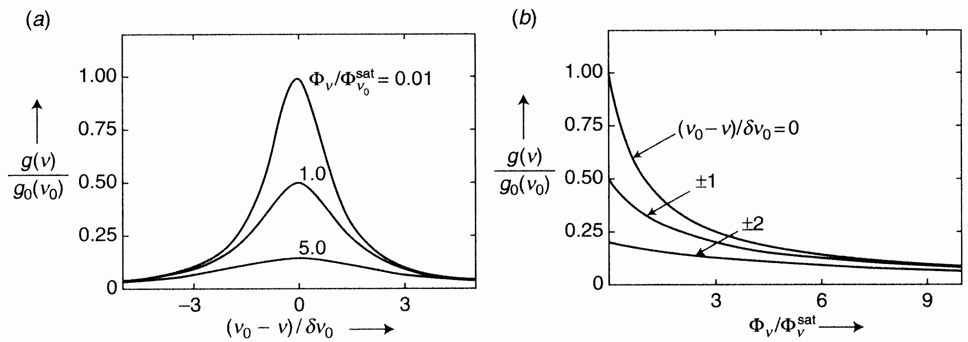

Figure 4.8 Saturated gain curves, according to Eq. (4.12.8).

same as the *power broadening* discussed in Section 4.11. The power-broadened gain bandwidth is the half width implied by (4.12.8), namely

$$\delta\nu_g = \delta\nu_0\sqrt{1 + \Phi_\nu/\Phi_{\nu_0}^{\text{sat}}}. \tag{4.12.9}$$

It is greater than the small-signal gain bandwidth and is seen to be exactly the width $\delta\nu_0'$ given in (4.11.6), as it should be, if we put $P = 0$ and $\Gamma_{21} = A_{21}$ in the expression for Φ_ν^{sat}.

In the case of a four-level laser we can obtain in a similar fashion a gain with the flux dependence (4.12.3), but with a saturation flux twice that given by (4.12.5) (Problem 4.8). The same basic flux dependence is also obtained when we include degeneracy and refractive index corrections to the gain coefficient (Section 3.12).

Three- and four-level lasers are idealizations that are seldom fully realized in practice. The gain–saturation formulas (4.12.3) and (4.12.8) are, however, applicable to a wide variety of actual lasers. That is, although (4.12.3) may be derived from simple models, it often applies outside the range of validity of these models. It is the most commonly assumed formula for the intensity dependence of the gain on a homogeneously broadened laser transition. In Section 4.14 we will consider the case of an inhomogeneously broadened transition.

In Eq. (4.12.5) P and Γ_{21} are the "decay rates" of the lower and upper laser levels, respectively [cf. Eqs. (4.7.4)]. The larger the decay rates, the larger the saturation flux. This makes good sense physically, for the larger the decay rates, the larger must be the stimulated emission rate necessary to saturate the transition, that is, to equalize the population densities $\overline{N}_1$ and $\overline{N}_2$. In fact the saturation flux (4.12.5) for a three-level laser is just the intensity for which the stimulated emission rate is the average of the upper- and lower-level decay rates (Problem 4.8). In general, the larger these decay rates, the larger the saturation flux. Equation (4.12.5) is an example of this general result.

In most cases of practical interest the pump rate P is small compared to Γ_{21} in a three-level laser and to Γ_{21} and Γ_{10} in a four-level laser. Then

$$I_\nu^{\text{sat}} \cong \frac{h\nu\Gamma_{21}}{2\sigma(\nu)} \quad \text{(three-level laser)} \tag{4.12.10}$$

and (Problem 4.8)

$$I_\nu^{\text{sat}} \cong \frac{h\nu\Gamma_{21}}{\sigma(\nu)} \quad \text{(four-level laser)}. \tag{4.12.11}$$

The relation $\sigma(\nu) \propto B_{21}$ [recall (3.12.18)] means that the saturation flux (4.12.5) is inversely proportional to the Einstein B coefficient for stimulated emission. This is another general result and is hardly surprising because the smaller B_{21} is, the greater the intensity necessary to achieve a given stimulated emission rate. For a Lorentzian lineshape function, (4.12.5) also predicts that the line-center saturation flux is directly proportional to the transition linewidth $\delta\nu_0$:

$$\Phi_{\nu_0}^{\text{sat}} = \frac{P + \Gamma_{21}}{2\sigma(\nu_0)} = \frac{4\pi^2\delta\nu_0}{\lambda^2 A}(P + \Gamma_{21}). \tag{4.12.12}$$

This too is a general conclusion that is applicable beyond the three- and four-level models.

The most important results of this section are Eqs. (4.12.3) and (4.12.8). We have obtained these results for the specific case of an ideal three-level laser, but we have emphasized that they apply to a large variety of real lasers under conditions of homogeneous line broadening. Whereas the detailed equations for the small-signal gain and saturation intensity are specific to the particular laser under consideration, the expressions (4.12.3) and (4.12.8) are more generally applicable. Indeed, it will usually be difficult to *calculate* g_0 and I_ν^{sat}, but we can often be confident nevertheless that the *form* of the intensity dependence of the gain described by (4.12.3) or (4.12.8) is correct.

We emphasize that these equations are applicable regardless of whether g_0 is positive (gain) or negative (absorption). That is, a medium may be saturated regardless of whether it is amplifying or absorbing. Thus, the absorption coefficient $a(\nu)$ of an absorbing medium will decrease as the intensity of the radiation is raised, as discussed in the preceding section. When the intensity is much larger than the line-center saturation intensity $I_{\nu_0}^{\text{sat}}$ characteristic of the medium, the absorption coefficient is very small [$a(\nu) \approx 0$], which means that the medium is practically transparent to high-intensity radiation. In this case the medium is sometimes said to be "bleached" because it no longer absorbs radiation that is resonant with one of its transition frequencies. What is happening in the case of such strong saturation is that the stimulated emission (and absorption) rate has become much greater than the decay rate of the *upper* level of the transition. An atom that has absorbed a photon will then be quickly induced to return to the lower level and give the photon back to the field by stimulated emission. This occurs, with high probability, before the absorbed energy can be dissipated as heat or fluorescence. Thus, no energy is lost by the incident field; the medium has been made effectively transparent ("bleached") by virtue of the high intensity of the field.

4.13 SPATIAL HOLE BURNING

In this section we will consider more carefully the meaning of the "intensity." Intensity refers to the electromagnetic energy flow per unit area per unit time, but in most lasers we have *standing* waves rather than traveling waves. The gain–saturation formulas (4.12.3) and (4.12.8) are often written with Φ_ν assumed to be the sum of the fluxes of the two traveling waves (Fig. 4.3):

$$\Phi_\nu \longrightarrow \Phi_\nu = \Phi_\nu^{(+)} + \Phi_\nu^{(-)}. \tag{4.13.1}$$

However, this is not quite correct, for it ignores the interference of the two traveling waves. The electromagnetic energy density u is proportional to the square of the electric field:[3]

$$u = \epsilon_0 \mathbf{E}^2(\mathbf{r}, t) = \epsilon_0 \mathbf{E}_0^2 \cos^2 \omega t \sin^2 kz. \tag{4.13.2}$$

We replace $\cos^2 \omega t$ by $\frac{1}{2}$, its average value over times long compared to an optical period $2\pi/\omega \approx 10^{-14}$ s, and write

$$u = \frac{\epsilon_0}{2} E_0^2 \sin^2 kz. \tag{4.13.3}$$

Now a cavity standing-wave field is the sum of two oppositely propagating traveling-wave fields:

$$\begin{aligned} E(z, t) &= E_0 \cos \omega t \sin kz = \tfrac{1}{2} E_0[\sin(kz - \omega t) + \sin(kz + \omega t)] \\ &= E_+(z, t) + E_-(z, t), \end{aligned} \tag{4.13.4}$$

where the two electric waves

$$E_\pm(z, t) = \tfrac{1}{2} E_0 \sin(kz \mp \omega t) \tag{4.13.5}$$

propagate in the positive $(+)$ and negative $(-)$ z directions. The time-averaged square of the electric field (4.13.5) gives a field energy density $u = u^{(+)} + u^{(-)}$, where

$$u^{(\pm)} = \frac{\epsilon_0}{8} E_0^2. \tag{4.13.6}$$

From (4.13.3) and (4.13.6), therefore, it follows that

$$\Phi_\nu \equiv \frac{c}{h\nu} u = 2\left[\Phi_\nu^{(+)} + \Phi_\nu^{(-)}\right] \sin^2 kz, \tag{4.13.7}$$

or, in terms of the intensity $I_\nu \equiv h\nu \Phi_\nu$,

$$I_\nu = 2\left[I_\nu^{(+)} + I_\nu^{(-)}\right] \sin^2 kz. \tag{4.13.8}$$

Thus, it is not correct to use (4.13.1) as the flux in the gain saturation formulas (4.12.3) and (4.12.8). We should use (4.13.7), which accounts properly for the

[3]For simplicity we assume in this section that the refractive index $n \approx 1$.

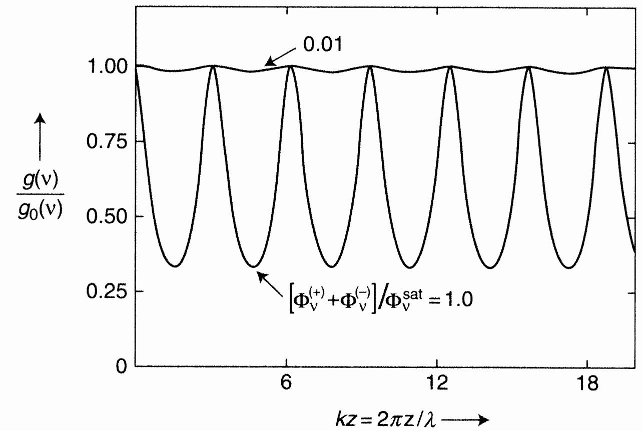

Figure 4.9 Spatial hole burning in the gain curve, according to Eq. (4.13.9).

interference of the two traveling-wave fields. Then the gain–saturation formula for a homogeneously broadened transition is

$$g(\nu) = \frac{g_0(\nu)}{1 + 2\left[\left(\Phi_\nu^{(+)} + \Phi_\nu^{(-)}\right)/\Phi_\nu^{\text{sat}}\right] \sin^2 kz}. \tag{4.13.9}$$

This saturation formula replaces (4.12.3) when the standing-wave nature of the cavity field is properly accounted for.

The $\sin^2 kz$ term in Eq. (4.13.9) gives rise to what is called *spatial hole burning* in the gain coefficient $g(\nu)$. At points z for which $\sin^2 kz = 0$, $g(\nu)$ takes on its maximum value, namely the small-signal value $g_0(\nu)$. Where $\sin^2 kz = 1$, however, $g(\nu)$ has its minimum value, that is, it is most strongly saturated; a "hole" is "burned" in the curve of $g(\nu)$ vs. z (Fig. 4.9). The holes in this curve are separated by $\Delta z = \pi/k = \lambda/2$. Thus, $g(\nu)$ varies with z on the scale of the laser wavelength.

This rapid variation of $g(\nu)$ with z suggests the approximation of replacing $\sin^2 kz$ by its spatial average, $\frac{1}{2}$, in Eq. (4.13.9). In this approximation we take

$$g(\nu) = \frac{g_0(\nu)}{1 + \left(\Phi_\nu^{(+)} + \Phi_\nu^{(-)}\right)/\Phi_\nu^{\text{sat}}}, \tag{4.13.10}$$

which is the result obtained by using (4.13.1) in the gain–saturation formula (4.12.3). This approximation ignores the spatial dependence of the intracavity field and, therefore, also the spatial hole burning of the gain coefficient. It is called the *uniform-field approximation* or the *spatial mean-field approximation.*

4.14 SPECTRAL HOLE BURNING

In an inhomogeneously broadened medium the different atoms have different central transition frequencies ν_0'. This may be due to their different velocities and the Doppler effect (Section 3.9), the presence of different isotopes having slightly different

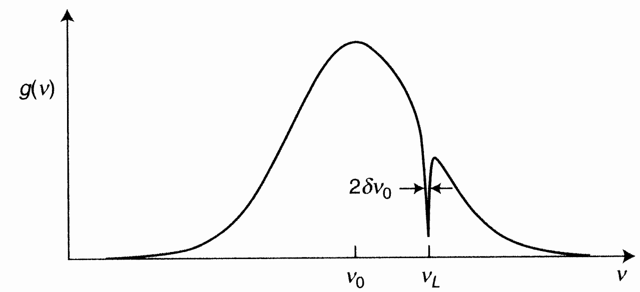

Figure 4.10 Spectral hole burning in an inhomogeneously broadened gain profile. Radiation of frequency ν_L saturates only a spectral packet of atoms with frequency $\nu \approx \nu_L$.

energy levels, spatially nonuniform electric and magnetic fields causing shifts in the energy levels, or a host of other effects.

Saturation of the absorption or gain coefficient is more complicated in the case of inhomogeneous broadening. Atoms with different central frequencies ν_0' will be saturated according to (4.11.5), but there is a *distribution* of resonance frequencies ν_0'. The absorption or gain coefficient is obtained by integrating the contributions from different frequency components, or *spectral packets*, each of which saturates to a different degree depending on its detuning from the cavity mode frequency ν.

Equation (4.11.5) implies that the absorption or gain is more strongly saturated for spectral packets with frequency $\nu_0' \approx \nu$; spectral packets with frequencies detuned from ν by much more than the homogeneous linewidth are hardly saturated at all. This is illustrated in Fig. 4.10. The selective saturation leads to *spectral hole burning* in the gain curve. The width of a hole is just the homogeneous linewidth $\delta\nu_0$ (if power broadening is small), while the depth is determined by the field intensity. When the field intensity is very large the hole "touches down," that is, the gain at the center of the hole is fully saturated.

Spectral hole burning is especially interesting in the case of a purely Doppler-broadened gain medium.[4] Suppose the cavity mode frequency ν is different from the center frequency ν_0 of the Doppler gain profile. Consider, for example, the traveling-wave field propagating in the positive z direction. This wave will strongly saturate the spectral packet of atoms with Doppler-shifted frequencies $\nu_0' = \nu$; the Doppler effect has brought these atoms into resonance with the wave. Therefore these atoms have the z component of velocity given by (cf. Section 3.9)

$$\nu = \nu_0' = \nu_0\left(1 + \frac{v}{c}\right), \tag{4.14.1a}$$

or

$$\frac{v}{c} = \frac{\nu - \nu_0}{\nu_0}, \tag{4.14.1b}$$

where ν_0 is the resonance frequency of a stationary atom. If $\nu > \nu_0$, then v is positive. This means that the atoms that have been Doppler shifted into resonance are moving in

[4]Spectral hole burning is the origin of the *Lamb dip* in Doppler-broadened lasers, as discussed in Section 5.8.

the positive z direction, the same direction in which the traveling wave is propagating. If $\nu < \nu_0$, on the other hand, v/c is negative, or in other words the atoms must be moving in the negative z direction, opposite to the traveling wave, in order to be shifted into resonance.

Consider the saturation of the gain coefficient when there is Doppler broadening. For simplicity we will assume a single traveling wave of frequency ν. With atoms of velocity v along the z direction we associate the saturated gain coefficient

$$\begin{aligned} g(\nu, v) &= \frac{\lambda^2 A_{21}}{8\pi}\frac{P - \Gamma_{21}}{P + \Gamma_{21}} N_T \frac{\delta\nu_0/\pi}{(\nu_0 - \nu + \nu_0 v/c)^2 + \delta\nu_g^2} \\ &\equiv C\frac{\delta\nu_0/\pi}{(\nu_0 - \nu + \nu_0 v/c)^2 + \delta\nu_g^2} \end{aligned} \tag{4.14.2}$$

that follows from the three-level laser model (Section 4.12) with the Doppler shift included in the (power-broadened) Lorentzian lineshape function. We now calculate the total gain coefficient due to atoms of all velocities by integrating over the Maxwell–Boltzmann distribution for atoms of molecular weight M_X, exactly as in Section 3.9:

$$\begin{aligned} g(\nu) &= \int_{-\infty}^{\infty} dv \left(\frac{M_X}{2\pi RT}\right)^{1/2} e^{-M_X v^2/2RT} g(\nu, v) \\ &= C\left(\frac{M_X}{2\pi RT}\right)^{1/2} \frac{\delta\nu_0}{\pi}\int_{-\infty}^{\infty} \frac{dv\, e^{-M_X v^2/2RT}}{(\nu_0 - \nu + \nu_0 v/c)^2 + \delta\nu_g^2} \\ &= C\frac{4\ln 2}{\pi^{3/2}}\frac{\delta\nu_0}{\delta\nu_D^2}\int_{-\infty}^{\infty}\frac{dy\, e^{-y^2}}{(x+y)^2 + b'^2}, \end{aligned} \tag{4.14.3}$$

where we have made the same change of variables as in Section 3.10 and defined x as in Eq. (3.10.4); b' is defined by replacing $\delta\nu_0$ by $\delta\nu_g$ in (3.10.5):

$$b' = (4\ln 2)^{1/2}\frac{\delta\nu_g}{\delta\nu_D}. \tag{4.14.4}$$

As in Section 3.10 it is convenient to consider the case $x = 0$, the case where the field frequency ν exactly matches the central frequency ν_0 of the Doppler lineshape. Then, using (3.10.7),

$$g(\nu) = C\frac{4\ln 2}{\pi^{3/2}}\frac{\delta\nu_0}{\delta\nu_D^2}\frac{\pi}{b'}e^{b'^2}\operatorname{erfc}(b') = C\left(\frac{4\ln 2}{\pi}\right)^{1/2}\frac{\delta\nu_0}{\delta\nu_D\delta\nu_g}e^{b'^2}\operatorname{erfc}(b'). \tag{4.14.5}$$

Suppose first that the transition is Doppler broadened and the intensity of the field is well below saturation, so that $\delta\nu_D \gg \delta\nu_0$, $\delta\nu_g \approx \delta\nu_0$, and $b' \approx (4\ln 2)^{1/2}\delta\nu_0/\delta\nu_D \ll 1$. Then, from (3.10.12), we obtain

$$g(\nu) \approx C\left(\frac{4\ln 2}{\pi}\right)^{1/2}\frac{1}{\delta\nu_D} = g_0^{(D)}(\nu_0), \tag{4.14.6}$$

where $g_0^{(D)}(\nu_0)$ is the small-signal gain at $\nu = \nu_0$ for Doppler broadening. If $\delta\nu_D \gg \delta\nu_g$ but $\delta\nu_g$ differs significantly from $\delta\nu_0$, then $b' \ll 1$ and we can still use (3.10.13), but now

$$g(\nu) \approx C\left(\frac{4\ln 2}{\pi}\right)^{1/2} \frac{\delta\nu_0}{\delta\nu_D \delta\nu_g} = C\left(\frac{4\ln 2}{\pi}\right)^{1/2} \frac{1}{\delta\nu_D} \frac{1}{\sqrt{1 + I_{\nu_0}/I_{\nu_0}^{\text{sat}}}}$$
$$= \frac{g_0^{(D)}(\nu_0)}{\sqrt{1 + I_{\nu_0}/I_{\nu_0}^{\text{sat}}}}. \tag{4.14.7}$$

Thus, a Doppler-broadened transition saturates not as $1/[1 + I_{\nu_0}/I_{\nu_0}^{\text{sat}}]$ but as $1/\sqrt{1 + I_{\nu_0}/I_{\nu_0}^{\text{sat}}}$. This is true so long as $\delta\nu_g$ is not too large: if $\delta\nu_g \gg \delta\nu_D$, then the approximation (3.10.10) [with b' replacing b] is applicable and (4.14.5) becomes

$$g(\nu) \approx C\left(\frac{4\ln 2}{\pi}\right)^{1/2} \frac{\delta\nu_0}{\delta\nu_D \delta\nu_g} \frac{1}{\pi^{1/2}} \frac{\delta\nu_D}{(4\ln 2)^{1/2}\delta\nu_g} = C\frac{1}{\pi}\frac{\delta\nu_0}{\delta\nu_g^2}$$
$$= C\frac{1/(\pi\delta\nu_0)}{1 + I_{\nu_0}/I_{\nu_0}^{\text{sat}}} = \frac{g_0^{(H)}(\nu_0)}{1 + I_{\nu_0}/I_{\nu_0}^{\text{sat}}}, \tag{4.14.8}$$

where $g_0^{(H)}(\nu_0)$ is the line-center small-signal gain for the homogeneously broadened transition with Lorentzian HWHM $\delta\nu_0$. In this power-broadened limit we recover the factor $1/[1 + I_{\nu_0}/I_{\nu_0}^{\text{sat}}]$ characteristic of *homogeneous* broadening.

Inhomogeneously broadened lasers, and in particular Doppler-broadened lasers, are generally much more complicated to treat theoretically than homogeneously broadened lasers, especially if counterpropagating traveling waves and spatial hole burning are taken into account.

4.15 SUMMARY

In Chapter 1 we gave a very crude description of laser action and introduced some fundamental concepts such as gain and threshold. In the intervening chapters we have gone more deeply into the theory of the interaction of light and matter, and in the present chapter we have begun to apply what we have learned to laser theory.

The most important theoretical tools for our understanding of lasers are the rate equations for level populations and field intensities. These equations generally include effects of pumping, collisions, absorption, spontaneous and stimulated emission, field gain and loss, and other processes that may be pertinent for a particular laser. We have used such equations to discuss three- and four-level lasers, and in fact the use of rate equations will be a dominant theme of the following chapters. We have also discussed the concept of saturation which, as we will see in the next chapter, is a major consideration in determining how much output power can be obtained with a given laser.

PROBLEMS

4.1. Show how (3.12.5) and (4.3.4a) are modified if the light propagates toward $-z$ rather than $+z$, and derive (4.3.4b).

4.2. **(a)** Solve (4.5.7) as a function of time for the (unusual) case that the equation's loss parameters satisfy $\Gamma_{21} + A_{21} = (c/2L)(1 - r_1 r_2)$. Give the steady-state value of $n_2 + q_\nu$.

(b) Find the steady-state solution for q_ν in terms of p and n_2 for arbitrary loss parameters.

4.3. **(a)** Write the full set of equations for a three-level laser by modifying (4.7.4) and including the following equation for the third level (as shown in Fig. 4.6): $dN_3/dt = PN_1 - \Gamma_{32}N_3$, and show that the full set of equations satisfies $N_1 + N_2 + N_3 = \text{constant} = N_T$.

(b) Determine the steady-state values of the three level populations.

(c) Find the condition under which it is satisfactory to neglect level 3 [$N_3 \approx 0$] and to use Eqs. (4.7.2) and (4.7.4) as written in the text.

4.4. Solve Eqs. (4.8.1) for the steady-state value of $\overline{N}_2 - \overline{N}_1$, and show that (4.8.4) gives the limiting value as the stimulated emission rate decreases to zero.

4.5. Estimate the density of chromium ions in ruby, assuming that the concentration of chromium in ruby is about 0.05% by weight.

4.6. **(a)** Calculate the transition dipole moment $e|\mathbf{x}_{12}|$ for the $3S_{1/2}(F = 2, M = 2) \leftrightarrow 3P_{3/2}(F = 3, M = 3)$ transition of sodium.

(b) What is the oscillator strength of this transition?

4.7. Derive the formula analogous to (4.12.3) for a four-level laser, and write the expression for the saturation intensity.

4.8. Show that the saturation intensity I_ν^{sat} of a three-level laser [Eq. (4.12.6)] is the intensity for which the stimulated emission rate is the average of the upper- and lower-level decay rates of the laser transition. Find the corresponding expression that follows from laser equations (4.5.2).

4.9. **(a)** A gain cell of length 10 cm has a small-signal gain coefficient of 0.025 cm^{-1}. Two mirrors having the same reflectivity are placed at the ends of the cell. Assuming that scattering losses are negligible, calculate the reflectivity necessary for lasing.

(b) If the gain curve has a FWHM width of 1 GHz, what is the maximum number of modes that can lase?